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A NOTE ON POWERS OF HAUSDORFF FIELDS

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1. Statement of results

The following duality theorem does not seem to have been explicitly noted. Let K be a hausdorff commutative ring, in the sense that the ring-operations are continuous. Let $\mathbf{A}$ be the category of hausdorff K-modules – in the sense that addition and scalar multiplication in the modules are continuous – and continuous linear maps. Let $\mathbf{V}$ be the category of untopologized K-modules and linear maps, and let $T:\mathbf{A}^{\mathrm{op}}\to\mathbf{V}$ be the functor sending $A\in\mathbf{A}$ to the K-module $\mathbf{A}(A,K)$. It is easily seen that T has the left adjoint S sending $V\in\mathbf{V}$ to the K-module $\mathbf{V}(V,K)$, topologized as a subspace (obviously closed) of the power $K^{|V|}$ in $\mathbf{A}$, where $|V|$ is the underlying set of V. Since S takes its values in the (epireflective) full subcategory $\mathbf{SP}$ of $\mathbf{A}$ given by the closed submodules of powers K^X of K with $X\in\mathbf{Set}$, we have an adjunction $S\dashv R:\mathbf{SP}^{\mathrm{op}}\to\mathbf{V}$ where R is the restriction of T.

Theorem. *If the underlying ring of K is a field, then the adjunction $S\dashv R:\mathbf{SP}^{\mathrm{op}}\to\mathbf{V}$ is an equivalence of categories. If K is a discrete principal ideal domain, then R is fully faithful and and thus gives an equivalence between $\mathbf{SP}^{\mathrm{op}}$ and a full reflective subcategory of $\mathbf{V}$. In both cases $R:\mathbf{SP}^{\mathrm{op}}\to\mathbf{V}$ preserves arbitrary coproducts; in fact $T:\mathbf{A}^{\mathrm{op}}\to\mathbf{V}$ preserves these.*

Corollary 1. *If the hausdorff ring K is a field, every closed subspace of a power K^X of K is isomorphic to a power K^Y of K.* □

Corollary 2. *If K is a discrete principal ideal domain, and if $K^X\to B$ is an epimorphism in $\mathbf{SP}$, then B is a power K^Y of K.* □

Remarks. For discrete fields the Theorem is contained in Lefschetz [3]. When $K=\mathbf{R}$ or $\mathbf{C}$, Corollary 1 is contained in Exercise 6 of Chapter 4 of Schaefer [6], and is attributed to Martineau [5]. This Corollary for $K=\mathbf{R}$ was also rediscovered in [1].

As is clear from the proof, the analogous result still holds where K is a hausdorff division ring.

2. Proof of the theorem

We first verify that T sends arbitrary products in **A** to coproducts in **V**. For finite products this is clear. The general case reduces to this by the observation that the kernel of a map $g:\prod_{\lambda\in\Lambda}A_\lambda\to K$ contains $\prod_{\lambda\in M}A_\lambda$ for some $M\subset\Lambda$ with $\Lambda-M$ finite; we have only to consider $g^{-1}(W)$, where W is a neighbourhood of 0 in K which contains no ideal except $\{0\}$.

Now observe that the full subcategory **T** of **A** determined by the *finite* powers of K is the *Lawvere theory* [2] of K-modules, so that **V** is equivalent to the full reflective subcategory of the functor-category [**T**, **Set**] given by the finite-product-preserving functors; and observe that the composite of $T:\mathbf{A}^{\mathrm{op}}\to\mathbf{V}$ with the inclusion $\mathbf{V}\to[\mathbf{T},\mathbf{Set}]$ sends A to the representable functor $\mathbf{A}(A,-):\mathbf{T}\to\mathbf{Set}$. In consequence, as we may see from Ch. X, §5 of [4], the functor $ST:\mathbf{A}\to\mathbf{A}$ is the right Kan extension of the inclusion $J:\mathbf{T}\to\mathbf{A}$ along itself; so that STA is the limit in **A** of the functor $A/J\to\mathbf{A}$ sending the object $f:A\to K^n$ of the comma-category A/J to K^n, while the counit ε of the adjunction $S\dashv T$ has as its A-component the evident map $\varepsilon_A:A\to STA$ in **A** into this limit.

Using the canonical diagonal form for m-by-n matrices over the principal ideal domain K, and the fact that ideals of K are closed, we easily see that any submodule of K^n for a finite n is closed, and is isomorphic to K^m for some $m\le n$. It follows that every $A\to K^n$ in A/J factorizes as $A\to K^m\to K^n$ into a surjection followed by the inclusion of a closed submodule. Thus the cofiltered category A/J has, as an *initial* full subcategory (see [4], Ch. IX, §3), the codirected poset $A/\!\!/J$ of surjections $A\to K^n$. It follows that STA is equally the limit of $A/\!\!/J\to\mathbf{A}$. Since each map $A\to K^n$ in $A/\!\!/J$ is surjective and *a fortiori* has a dense image, the map $\varepsilon_A:A\to STA$ into the codirected limit also has a dense image, by a simple argument.

We conclude that ε_A is an isomorphism exactly when it is the inclusion of a closed subspace. When this is so, we have $A\in\mathbf{SP}$ since $STA\in\mathbf{SP}$. Conversely, when $A\in\mathbf{SP}$, so that we have a closed-subspace-inclusion $j:A\to K^X$, we observe that K^X is $S(X\cdot K)$, where $X\cdot K$ is the copower (the coproduct in **V** of X copies of K); hence j factorizes through ε_A, which is consequently a closed-subspace-inclusion. It follows that ε_A is invertible exactly when $A\in\mathbf{SP}$; so that $R:\mathbf{SP}^{\mathrm{op}}\to\mathbf{V}$ is fully faithful.

When K is a field, R is essentially surjective and is hence an equivalence; for any $V\in\mathbf{V}$, being a copower $X\cdot K$, is isomorphic to $R(K^X)$, by the first paragraph of the proof.

References

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